

TWO MEASURES ON CANTOR SETS

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ABSTRACT. We give an example of Cantor type set for which its equilibrium measure and the corresponding Hausdorff measure are mutually absolutely continuous. Also we show that these two measures are regular in Stahl-Totik sense.

1. INTRODUCTION

The relation between the α dimensional Hausdorff measure Λ_α and the harmonic measure ω on a finitely connected domain Ω is understood well. Due to Makarov [5], we know that, for a simply connected domain, $\dim \omega = 1$ where $\dim \omega := \inf\{\alpha : \omega \perp \Lambda_\alpha\}$. Pommerenke [8] gives a full characterization of parts of $\partial\Omega$ where ω is absolutely continuous or singular with respect to a linear Hausdorff measure. Later similar facts were obtained for finitely connected domains. In the infinitely connected case there are only particular results. Model example here is $\Omega = \overline{\mathbb{C}} \setminus K$ for a Cantor-type set K . On most of such sets we have the strict inequality $\dim \omega < \alpha_K$ (see, e.g. [1], [6], [12], [13]), where α_K stands for the Hausdorff dimension of K . This inequality implies that $\Lambda_{\alpha_K} \perp \omega$ on K . These results motivate the problem to find a Cantor set for which its harmonic measure and the corresponding Hausdorff measure are not mutually singular.

Recall that, for a dimension function h , a set $E \subset \mathbb{C}$ is an h -set if $0 < \Lambda_h(E) < \infty$ where Λ_h is the Hausdorff measure corresponding to the function h . We consider introduced in [4] Cantor-type sets $K(\gamma)$. In section 2 we present a function h that makes $K(\gamma)$ an h -set. In section 3 we show that Λ_h and ω are mutually absolutely continuous for $K(\gamma)$. In the last section we prove that these two measures are regular in Stahl-Totik sense.

We will denote by \log the natural logarithm, $Cap(\cdot)$ stands for the logarithmic capacity.

2. DIMENSION FUNCTION OF $K(\gamma)$

A function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a dimension function if it is increasing, continuous and $h(0) = 0$. Given set $E \subset \mathbb{C}$, its h -Hausdorff measure is defined as

$$(2.1) \quad \Lambda_h(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum h(r_j) : E \subset \bigcup B(z_j, r_j) \text{ with } r_j \leq \delta \right\},$$

where $B(z, r)$ is the open ball of radius r centered at z .

For the convenience of the reader we repeat the relevant material from [4]. Given sequence $\gamma = (\gamma_s)_{s=1}^\infty$ with $0 < \gamma_s \leq \frac{1}{32}$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$ and $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ for $s \in \mathbb{N}$. Consider the set

$$E_s := \{x \in \mathbb{R} : P_{2^{s+1}}(x) \leq 0\} = \bigcup_{j=1}^{2^s} I_{j,s}.$$

The s -th level *basic* intervals $I_{j,s}$ with lengths $l_{j,s}$ are disjoint and $\max_{1 \leq j \leq 2^s} l_{j,s} \rightarrow 0$ as $s \rightarrow \infty$. Since $E_{s+1} \subset E_s$, we have a Cantor type set $K(\gamma) := \cap_{s=0}^{\infty} E_s$. The set $K(\gamma)$ is non polar if and only $\sum_{s=1}^{\infty} 2^{-s} \log \frac{1}{\gamma_s} < \infty$. In this paper we make the assumption

$$(2.2) \quad \sum_{s=1}^{\infty} \gamma_s < \infty.$$

Let $M := 1 + \exp(16 \sum_{s=1}^{\infty} \gamma_s)$, so $M > 2$, and $\delta_s := \gamma_1 \gamma_2 \dots \gamma_s$. By Lemma 6 in [4],

$$(2.3) \quad \delta_s < l_{j,s} < M \cdot \delta_s \text{ for } 1 \leq j \leq 2^s.$$

We construct a dimension function for $K(\gamma)$, following Nevanlinna [7]. Let $\eta(\delta_s) = s$ for $s \in \mathbb{Z}_+$ with $\delta_0 := 1$. We define $\eta(t)$ for (δ_{s+1}, δ_s) by

$$\eta(t) = s + \frac{\log \frac{\delta_s}{t}}{\log \frac{\delta_s}{\delta_{s+1}}}.$$

This makes η continuous and monotonically decreasing on $(0, 1]$. In addition, we have $\lim_{t \rightarrow 0} \eta(t) = \infty$. Also observe that, for the derivative of η on (δ_{s+1}, δ_s) , we have

$$\frac{d\eta}{dt} = \frac{-1}{t \log \frac{1}{\gamma_{s+1}}} \geq \frac{-1}{t \log 32} \quad \text{and} \quad \frac{d\eta}{d \log t} \geq \frac{-1}{\log 32}.$$

Define $h(t) = 2^{-\eta(t)}$ for $0 < t \leq 1$ and $h(t) = 1$ for $t > 1$. Then h is a dimension function with $h(\delta_s) = 2^{-s}$ and

$$\frac{d \log h}{d \log t} \leq \frac{\log 2}{\log 32} < 1.$$

Therefore if $m > 1$ and $r \leq 1$ we get the following inequality:

$$\log \frac{h(r)}{h\left(\frac{r}{m}\right)} < \int_{r/m}^r d \log t = \log m.$$

Finally, we obtain

$$(2.4) \quad h(r) < m \cdot h\left(\frac{r}{m}\right) \text{ for } m > 1 \text{ and } 0 < r \leq 1.$$

Let us show that $K(\gamma)$ is an h -set for the given function h .

Theorem 2.1. *Let γ satisfy (2.2). Then $1/8 \leq \Lambda_h(K(\gamma)) \leq M/2$.*

Proof. First, observe that, by (2.3), for each $s \in \mathbb{N}$ the set $K(\gamma)$ can be covered by 2^s intervals of length $M \cdot \delta_s$. Since $M/2 > 1$, we have by (2.4),

$$\Lambda_h(K(\gamma)) \leq \limsup_{s \rightarrow \infty} (2^s \cdot h(M/2 \cdot \delta_s)) \leq \limsup_{s \rightarrow \infty} (2^s \cdot M/2 \cdot h(\delta_s)) = M/2.$$

We proceed to show the lower bound. Let (J_ν) be an open cover of $K(\gamma)$. Then, by compactness, there are finitely many intervals $(J_\nu)_{\nu=1}^m$ that cover $K(\gamma)$. Since $K(\gamma)$ is totally disconnected, we can assume that these intervals are disjoint. Each J_ν contains a closed subinterval $J'_\nu = [a_\nu, b_\nu]$ whose endpoints belong to $K(\gamma)$ and covers all points of $K(\gamma)$ in J_ν . Since the intervals $(J'_\nu)_{\nu=1}^m$ are disjoint, all a_ν, b_ν are endpoints of some basic intervals. Let n be the minimal number such that all $(a_\nu)_{\nu=1}^m, (b_\nu)_{\nu=1}^m$ are the

endpoints of n -th level. Thus, each $I_{j,n}$ for $1 \leq j \leq 2^n$ is contained in some J'_ν . Let N_ν be the number of n -th level intervals in J'_ν . Clearly, $\sum_{\nu=1}^m N_\nu = 2^n$.

For a fixed $\nu \in \{1, 2, \dots, m\}$, let q_ν be the smallest number such that J'_ν contains at least one basic interval I_{j,q_ν} . Clearly, $q_\nu \leq n$ and $l_{j,q_\nu} \leq d_\nu$ where d_ν is the length of J_ν . Therefore, by (2.3),

$$h(d_\nu) \geq h(l_{j,q_\nu}) \geq h(\delta_{q_\nu}) = 2^{-q_\nu}.$$

Let us cover J'_ν by the smallest set G_ν which is a finite union of adjacent intervals of the level q_ν . Observe that G_ν consists of at least one and at most four such intervals. Each interval of the q_ν -th level contains 2^{n-q_ν} subintervals of the n -th level. This gives at most $2^{n-q_\nu+2}$ intervals of level n in the set G_ν . Hence

$$N_\nu \leq 2^{n-q_\nu+2}.$$

Therefore,

$$\sum_{\nu=1}^m h(d_\nu) \geq \sum_{\nu=1}^m 2^{-q_\nu} \geq 2^{-n-2} \sum_{\nu=1}^m N_\nu = 1/4.$$

Since $h(d) < 2 \cdot h(d/2)$ from (2.4), finally we obtain the desired bound. \square

Similar arguments apply to the case of a part of $K(\gamma)$ on any basic interval.

Corollary 2.2. *Let γ satisfy (2.2). Then $2^{-s-3} \leq \Lambda_h(K(\gamma) \cap I_{j,s}) \leq M \cdot 2^{-s-1}$ for each $s \in \mathbb{N}$ and $1 \leq j \leq 2^s$.*

REMARK. A set E is called *dimensional* if there is at least one dimension function h that makes E an h -set. It should be noted that not all sets are dimensional. If we replace the condition $h(0) = 0$ by $h(0) \geq 0$, then any sequence gives a trivial example of a dimensionless set. Best in [2] presented an example of a dimensionless Cantor set provided $h(0) = 0$. The author considered dimension functions with the additional condition of concavity, but did not use it in his construction.

3. HARMONIC MEASURE AND HAUSDORFF MEASURE FOR $K(\gamma)$

Suppose we are given a non polar compact set K that coincides with its exterior boundary. Then for the equilibrium measure μ_K on K we have the representation $\mu_K(\cdot) = \omega(\infty, \cdot, \overline{\mathbb{C}} \setminus K)$ in terms of the value of the harmonic measure at infinity (see e.g. [10], T.4.3.14). Moreover, since measures $\omega(z_1, \cdot, \overline{\mathbb{C}} \setminus K)$ and $\omega(z_2, \cdot, \overline{\mathbb{C}} \setminus K)$ are mutually absolutely continuous (see e.g. [10] Cor.4.3.5), our main result is valid even if, instead of $\mu_{K(\gamma)}$, we take the measure corresponding to the value of the harmonic measure at any other point.

The next theorem follows immediately from the definition of Λ_h . It is a simple part of Frostman's theorem (see e.g. T.D.1 in [3]).

Theorem 3.1. *Let h be a dimension function. If μ is a positive Borel measure such that*

$$\mu(B(z, r)) \leq h(r)$$

for all z and r , then the following is valid for any Borel set E

$$\mu(E) \leq \Lambda_h(E).$$

For the converse relation we use a simple version of T.7.6.1.(a) in [9]. Here, $b(1)$ is the Besicovitch covering number corresponding to the line (one can take $b(1) = 5$).

Theorem 3.2. *Assume that μ is a Borel probability measure on \mathbb{R} and A is a bounded Borel subset of \mathbb{R} . If there exists a constant C such that*

$$h(r) \leq C \cdot \mu(B(x, r))$$

for all $x \in A$ and $r > 0$, then for any Borel set $E \subset A$

$$\Lambda_h(E) \leq b(1) C \cdot \mu(E).$$

The set $K(\gamma)$ is weakly equilibrium in the following sense. Given $s \in \mathbb{N}$, we uniformly distribute the mass 2^{-s} on each $I_{j,s}$ for $1 \leq j \leq 2^s$. Let us denote by λ_s the normalized in this sense Lebesgue measure on E_s , so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

Theorem 3.3. ([4], T.4) *Suppose $K(\gamma)$ is not polar. Then λ_s is weak star convergent to the equilibrium measure $\mu_{K(\gamma)}$.*

Corollary 3.4. *Suppose $K(\gamma)$ is not polar. Then $\mu_{K(\gamma)}(I_{j,s}) = 2^{-s}$ for each $s \in \mathbb{N}$ and $1 \leq j \leq 2^s$.*

Proof. Indeed, the characteristic function $\chi_{I_{j,s}}$ is continuous on E_n for $n \geq s$, where E_n is given in the construction of $K(\gamma)$. Therefore, $\mu_{K(\gamma)}(I_{j,s}) = \int \chi_{I_{j,s}} d\mu_{K(\gamma)} = \lim_{n \rightarrow \infty} \int \chi_{I_{j,s}} d\lambda_n = 2^{-s}$. \square

In our main theorem and below, by Λ_h we mean restricted to the compact set $K(\gamma)$ the Hausdorff measure corresponding to the constructed function h .

Theorem 3.5. *Let γ satisfy (2.2) and $K(\gamma)$ be non polar. Then measures $\mu_{K(\gamma)}$ and Λ_h are mutually absolutely continuous.*

Proof. Let us fix any open interval I of length $2r$ and show that

$$(3.1) \quad \mu_{K(\gamma)}(I) \leq 8 h(r).$$

Then, by Theorem 3.1, $\mu_{K(\gamma)}(E) \leq 8 \Lambda_h(E)$ for any Borel set E and $\mu_{K(\gamma)} \ll \Lambda_h$.

First suppose that the endpoints of I do not belong to $K(\gamma)$. Then there exists $I' = [a, b] \subset I$ which contains all points in $K(\gamma) \cap I$. Let us take, as above, minimal n and q such that both a and b are the endpoints of n -th level and I' contains at least one basic interval $I_{j,q}$. All points in $K(\gamma) \cap I$ can be covered by 4 adjacent intervals of the level q and this cover G contains $4 \cdot 2^{s-q}$ intervals of the level s for $s \geq q$. Hence $\int \chi_G d\lambda_s = 4 \cdot 2^{-q}$. The characteristic function $\chi_{I'}$ is continuous on E_s for $s \geq n$. By Theorem 3.3,

$$4 \cdot 2^{-q} \geq \lim_{s \rightarrow \infty} \int \chi_{I'} d\lambda_s = \int \chi_{I'} d\mu_{K(\gamma)} = \mu_{K(\gamma)}(I).$$

On the other hand, I contains some basic interval $I_{j,q}$. Therefore $2r > l_{j,q}$ and, by (2.3),

$$h(2r) \geq h(l_{j,q}) \geq h(\delta_q) = 2^{-q}.$$

Combining these inequalities with (2.4) gives (3.1):

$$8 h(r) > 4 h(2r) \geq 4 \cdot 2^{-q} \geq \mu_{K(\gamma)}(I).$$

Now let us consider the case when at least one of the endpoints of $I = (z - r, z + r)$ is contained in $K(\gamma)$. Since the set is totally disconnected, we can take two real null sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ such that the endpoints of $I_n = (z - r - \alpha_n, z + r + \beta_n)$ do not belong to $K(\gamma)$ for each n . Arguing as above, we see that

$$\mu_{K(\gamma)}(I) \leq \mu_{K(\gamma)}(I_n) \leq 8h(r + \varepsilon_n) \leq 8(1 + \varepsilon_n/r)h(r),$$

where $\varepsilon_n = \max\{\alpha_n, \beta_n\}$. Since $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, we have (3.1) for this case as well.

We proceed to show that $\Lambda_h \ll \mu_{K(\gamma)}$. Let us fix $x \in K(\gamma)$ and $r > 0$. In order to use Theorem 3.2, let us show that

$$(3.2) \quad h(r) \leq 2M \cdot \mu_{K(\gamma)}(I),$$

where $I = (x - r, x + r)$. Clearly, it is enough to consider only $r < 1$. Let us fix two consecutive basic intervals containing our point: $x \in I_{i,s} \subset I_{j,s-1}$ with $l_{i,s} \leq r < l_{j,s-1}$. Then $I \supset I_{i,s}$ and $\mu_{K(\gamma)}(I) \geq 2^{-s}$, by Corollary 3.4. On the other hand, by (2.3) and (2.4),

$$h(r) < h(l_{j,s-1}) < h(M\delta_{s-1}) < Mh(\delta_{s-1}) = M2^{-s+1}.$$

This gives (3.2) and completes the proof. \square

Example 3.6. The sequence γ with $\gamma_s = \exp(-8s + 4)$ for $s \in \mathbb{N}$ satisfies all desired conditions. In particular, $\text{Cap}(K(\gamma)) = \exp(-12)$, so $K(\gamma)$ is not polar. Here, $\delta_s = \exp(-4s^2)$ and $\eta(t) = s + \frac{-4s^2 - \log t}{8s+4}$ for $\delta_s \leq t < \delta_{s-1}$. The Hausdorff measure Λ_h corresponding to the function $h = 2^{-\eta(t)}$ and $\mu_{K(\gamma)}$ are mutually absolutely continuous.

4. REGULARITY OF $\mu_{K(\gamma)}$ AND Λ_h IN STAHL-TOTIK SENSE

One of active directions of the theory of general orthogonal polynomials is the exploration of the case of non discrete measures that are singular with respect to the Lebesgue measure. Important class of *regular in Stahl-Totik sense* measures was introduced in [11] in the following way. Let μ be a finite Borel measure with compact support S_μ on \mathbb{C} . Then we can uniquely define a sequence of orthonormal polynomials $p_n(\mu; z) = a_n z^n + \dots$ with a positive leading coefficient a_n . By definition, $\mu \in \mathbf{Reg}$ if $\lim_{n \rightarrow \infty} a_n^{-\frac{1}{n}} = \text{Cap}(S_\mu)$. One of sufficient conditions of regularity was suggested in [11] by means of the set $A_\mu = \{z \in S_\mu : \limsup_{r \rightarrow 0^+} \frac{\log 1/\mu(\overline{B(z,r)})}{\log 1/r} < \infty\}$.

Theorem 4.1. (*T.4.2.1 in [11]*) *If $\text{Cap}(A_\mu) = \text{Cap}(S_\mu)$ then $\mu \in \mathbf{Reg}$.*

Let us show that, in our case, $A_\mu = S_\mu$ for both measures $\mu_{K(\gamma)}$ and Λ_h .

Theorem 4.2. *Let $K(\gamma)$ satisfy the conditions of Theorem 3.5. Then $\mu_{K(\gamma)}$ and Λ_h are regular in Stahl-Totik sense.*

Proof. Since $\Lambda_h(E) \geq \mu_{K(\gamma)(E)}/8$ for any Borel subset E of $K(\gamma)$, we only check the equilibrium measure. Let $z \in K(\gamma)$ and $r > 0$ be given. As in Theorem 3.5, fix s such that $z \in I_{i,s} \subset I_{j,s-1}$ with $l_{i,s} \leq r < l_{j,s-1}$. Then $I_{i,s} \subset \overline{B(z,r)}$. By Corollary 3.4,

$\mu(\overline{B(z, r)}) \geq \mu(I_{i, s}) = 2^{-s}$. On the other hand, $r < M \delta_{s-1} \leq M 32^{1-s}$ as $\gamma_k \leq 1/32$. Since $s \rightarrow \infty$ as $r \rightarrow 0^+$, we see that

$$\limsup_{r \rightarrow 0^+} \frac{\log 1/\mu(\overline{B(z, r)})}{\log 1/r} \leq 1/5,$$

which completes the proof. □

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